

Non-extensive statistics, relativistic kinetic theory and fluid dynamics

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Abstract. Experimental particle spectra can be successfully described by power-law tailed energy distributions characteristic to canonical equilibrium distributions associated to Rényi's or Tsallis' entropy formula - over a wide range of energies, colliding system sizes, and produced hadron sorts. In order to derive its evolution one needs a corresponding dynamical description of the system which results in such final state observables. The equations of relativistic fluid dynamics are obtained from a non-extensive Boltzmann equation consistent with Tsallis' non-extensive q -entropy formula. The transport coefficients like shear viscosity, bulk viscosity, and heat conductivity are evaluated based on a linearized collision integral.

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1 Introduction

Experimental particle spectra at the Relativistic Heavy-Ion Collider (RHIC) and at the Large Hadron Collider (LHC) can be successfully described by power-law tailed energy distributions characteristic to canonical equilibrium distributions associated to Rényi's or Tsallis' entropy formula - over a wide range of energies, colliding system sizes and produced hadron sorts [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Such a generalized statistical model can be solicited on the valence quark level, since several experimental findings on the azimuthal flow component v_2 [16, 17, 18] as well as on the non-extensivity measure $(q - 1)$ show quark number scaling [9, 10, 11, 12]. Therefore it is of primary importance to establish a theoretical tool for the description of the relativistic flow of such a quark matter.

The non-extensivity effects in general can be viewed as an effective measure of finite available phase-space [19, 20], either due to spatial finiteness of the reaction zone or due to long range entanglement of quark and gluon strings in the hadron formation process, representing a leading order deviation between microcanonical and canonical approximations in traditional terms [20, 21]. Taken a hadronizing quark matter with effective, massive and entangled quarks seriously, the $(q - 1)$ measure for mesons should be about half, for baryons and antibaryons about one third of the quark value [9]. It means that if the power-law tailed observed p_T -distribution of hadrons was performed practically in the quark phase, then the non-extensivity effects should have been even more pronounced on the quark than on the hadron level. Therefore it is unavoidable to test

those models, which are based on or at least are consistent with non-extensive thermodynamics, deriving their equilibrium state based on the Tsallis' or Rényi entropy.

An important step towards this goal is the development of a relativistic fluid dynamics, which is fully consistent with non-extensivity. Although initial steps were taken by considering the so-called relativistic perfect q -hydrodynamics [22, 23, 24], the relativistic equations of motion and transport coefficients for a finite q have only recently been derived [25].

In this paper we present the first full derivation of a consequent fluid dynamics from a non-extensive relativistic Boltzmann equation (NEBE). This equation retains the form but generalizes the well known Boltzmann transport equation, with a single parameter denoted by q . We derive the relativistic fluid-dynamical equations of motion from this non-extensive relativistic Boltzmann transport equation. Applying well known traditional methods, we calculate the transport coefficients like heat-conductivity, shear and bulk viscosity. We also analyse the question which entropy and conserved Noether four-currents have to be used in a proper, consistent non-extensive approach. Finally we also discuss Grad's method of moments to derive relativistic causal fluid dynamics in this framework.

2 Non-extensive statistics, kinetic and fluid dynamical equations

A non-extensive q -generalization of the Boltzmann-Gibbs (BG) entropy was proposed by C. Tsallis [19, 26] based on

the following formula,

$$S_q \equiv -k_B \sum_{i=1}^W (p_i)^q \ln_q(p_i) = \frac{k_B}{1-q} \left(\sum_{i=1}^W (p_i)^q - 1 \right), \quad (1)$$

where k_B is the Boltzmann constant, p_i is the probability of the i th from a total of W possible microstates, such that $p_i \in [0, 1]$ with the normalization condition $\sum_{i=1}^W p_i = 1$. Here the q -parameter is a real number $q \in [0, 2]$, and \ln_q is defined as the q -logarithmic function,

$$\ln_q(x) = \frac{x^{1-q} - 1}{1-q}, \quad (2)$$

for any $x > 0$. The inverse of the q -logarithmic function is the q -exponential,

$$\exp_q(x) \equiv e_q(x) = [1 + (1-q)x]^{1/(1-q)}, \quad (3)$$

for any $x > 1/(q-1)$. These functions return the well known natural logarithm and exponential functions for $q = 1$, that is $\ln_{q=1}(x) = \ln(x)$ and $\exp_{q=1}(x) = \exp(x)$.

For later reference we also define the following dual functions [27],

$$\ln_{q^*}(x) = \frac{x^{q-1} - 1}{q-1}, \quad (4)$$

$$e_{q^*}(x) = [1 + (q-1)x]^{1/(q-1)}, \quad (5)$$

and

$$\text{Ln}_q(x) = q \left(\frac{x^{q-1} - 1}{q-1} \right), \quad (6)$$

$$\text{E}_q(x) = \left[1 + \left(\frac{q-1}{q} \right) x \right]^{1/(q-1)}. \quad (7)$$

One can show using Eqs. (2,3) with $q^* = 2 - q$ that $\ln_{q^*}(x) = \ln_{2-q}(x)$, $e_{q^*}(x) = e_{2-q}(x)$. Similarly, $\text{E}_q(x) = e_{q^*}(x/q)$, $e_q(x) = \text{E}_{q^*}(q^*x)$ and $\text{Ln}_q(x) \equiv -q \ln_q(1/x) = q \ln_{q^*}(x)$. These relations are called dualities and using these functions we can re-write the q -generalized entropy in two further equivalent forms:

$$S_q \equiv -k_B \sum_{i=1}^W p_i \ln_{q^*}(p_i) = -\frac{k_B}{q} \sum_{i=1}^W p_i \text{Ln}_q(p_i). \quad (8)$$

We note that the Rényi formula, $S_R = -\frac{k_B}{q-1} \ln \sum_{i=1}^W p_i^q$ is preferred in order to simply fulfill the additivity property inherent in the zeroth law of thermodynamics [28]. Since the Rényi entropy is proportional to the logarithm of a phase-space integral, its local four-current density cannot be uniquely defined. Consequently, for deriving fluid dynamics and keeping the linear integration-operator structure of the Boltzmann equation, the Tsallis formula is the one which provides us a straightforward derivation.

Let us introduce the single particle phase-space distribution function $f_k = f(t, \mathbf{x}, k^0, \mathbf{k})$. The distribution function is normalized to the number of particles in the

system, hence in an invariant phase-space volume element $(\mathbf{x}, \mathbf{x} + d\mathbf{x}, \mathbf{k}, \mathbf{k} + d\mathbf{k})$ at time t , the number of particles is given by,

$$d\mathcal{N}(t) = f_k d^3x d^3k, \quad (9)$$

where $x = |\mathbf{x}|$ and $k = |\mathbf{k}|$. From now on we work with natural units and set $k_B = \hbar = c = 1$.

The q -generalized entropy from Eq. (1) can also be expressed with the help of the single particle distribution function. This leads to the following relativistic q -generalized entropy four-current,

$$S^\mu = - \int dK k^\mu [(f_k)^q \ln_q(f_k) - f_k]. \quad (10)$$

Here $dK = g d^3\mathbf{k} / ((2\pi)^3 k^0)$ is the invariant momentum-space volume with g denoting the number of internal degrees of freedom, such as the spin degeneracy. The timelike component of the relativistic entropy four-current $S^0 = - \int d^3\mathbf{k} (g/(2\pi)^3) [(f_k)^q \ln_q(f_k) - f_k]$ retains the familiar Tsallis entropy, while the remaining components define a three vector.

It is imperative to point out that in kinetic theory there are different definitions of the q -entropy four-current in use. These definitions are not equivalent with each other, and also lead to different thermodynamical relations as will be discussed later in this paper. For example, Lima and Silva et al. [29,30] applied $S_S^\mu = - \int \frac{d^3\mathbf{k}}{k^0} k^\mu f_k^q \ln_q f_k$, while Lavagno [31,32] used $S_L^\mu = - \int \frac{d^3\mathbf{k}}{k^0} k^\mu f_k^q [\ln_q f_k - 1]$. These alternative definitions did not explicitly denote the finite size of the elementary phase-space cell volume $(2\pi\hbar)^3$. The corrections for quantum statistics can be added, both for fermions ($z = 1$) and for bosons ($z = -1$). A correct q -generalized quantum statistical entropy was proposed by Cleymans and Worku recently [33,34,35],

$$S_Q^\mu = - \int dK k^\mu \left[f_k^q \ln_q f_k + \frac{1}{z} (1 - z f_k)^q \ln_q (1 - z f_k) \right]. \quad (11)$$

The $z = 0$ limit corresponds to Eq. (10) and was used in this form by Osada and Wilk [22,23,24].

The second law of thermodynamics demands positive entropy production,

$$\partial_\mu S^\mu = - \int dK \ln_q(f_k) [k^\mu \partial_\mu (f_k)^q] \geq 0, \quad (12)$$

This entropy production formula suggests the following kinetic equation, first introduced by Lavagno [31], from now on referred to as NEBE,

$$k^\mu \partial_\mu \tilde{f}_k = C[f], \quad (13)$$

where $C[f]$ is the collision integral and we use the notation,

$$\tilde{f}_k = (f_k)^q. \quad (14)$$

This NEBE is postulated with the purpose to study classical statistical effects due to correlations possibly induced

by finite system size, compared to the characteristic interaction range.

We also stress here that this distribution determines a different number of particles from Eq. (9),

$$d\tilde{N}(t) = \tilde{f}_k d^3x d^3k. \quad (15)$$

Of course this should not mean that the total number of particles in a system would turn out to be different after integration over the whole phase-space.

The collision integral, $C[f]$, specifies the change in \tilde{f}_k due to binary collisions among particles with initial incoming momenta of k^μ , k'^μ and outgoing final momenta p^μ , p'^μ . It is constructed with the help of the invariant transition rate $W_{kk' \rightarrow pp'}$. To fulfill the detailed balance property, the transition rate has to be symmetric with respect to the sequence of final states $W_{kk' \rightarrow pp'} = W_{kk' \rightarrow p'p}$, as well as symmetric for time reversed processes $W_{kk' \rightarrow pp'} = W_{pp' \rightarrow kk'}$. The collision integral is formally written as,

$$C[f] = \frac{1}{2} \int dK' dP dP' W_{kk' \rightarrow pp'} \times (H_q[f_p, f_{p'}] - H_q[f_k, f_{k'}]), \quad (16)$$

where the q -generalized version of the assumption of molecular chaos, the so-called q -generalized Stosszahlansatz, is given as in [31,32]

$$H_q[f_k, f_{k'}] = \exp_q[\ln_q(f_k) + \ln_q(f_{k'})]. \quad (17)$$

This formula is a postulate for the correlation between two particles with different momenta before and after the collision. For $q = 1$ it returns the familiar collision integral where the number of binary collisions around space-time coordinates x^μ is proportional to $H_1[f_k, f_{k'}] = f_k f_{k'}$.

Clearly also other postulates can be made, which all satisfy an H-theorem [7]. The $H_q[f_k, f_{k'}]$ in the above formula, in general can be replaced by any kk' -symmetric functional. For example, using

$$\mathcal{H}_q[f_k, f_{k'}] = \exp_q[\ln_q(f_k) + \ln_q(f_{k'}) + (1-q)\ln_q(f_k)\ln_q(f_{k'})], \quad (18)$$

turns out to be the original Boltzmann type of factorization ansatz neglecting correlations, $\mathcal{H}_q[f_k, f_{k'}] = f_k f_{k'}$ even for $q \neq 1$. This however does not lead automatically to the classical kinetic theory, just then, if the k^μ four-momenta are treated additively in the collisions. In this paper we assume that this is the case.

Note that in the literature not only for the entropy but also for the NEBE collision kernel, slightly different versions of the kinetic equation and correlation function exist, involving the \ln_q [27] or \ln_{q^*} [29,30] functions. We return to discuss these later in the Appendix A.1.

The entropy production vanishes in local q -equilibrium that is, $\partial_\mu S^\mu \equiv - \int dK \ln_q(f_{0k}) C[f_0] = 0$. Therefore the distribution function which satisfies $\partial_\mu S^\mu[f_{0k}] = 0$ locally, is the q -equilibrium distribution. This follows assuming that $\ln_q(f_{0k}) = \alpha_0 - \beta_0^\mu k_\mu$ hence,

$$f_{0k} = \exp_q(\alpha_0 - \beta_0^\mu k_\mu). \quad (19)$$

Here α_0 and $\beta_0^\mu = \beta_0 u^\mu$ are collision invariants with u^μ being a four-vector normalized to one, $u^\mu u_\mu = 1$. These quantities are identified with the inverse temperature $\beta_0 = 1/T$, the chemical potential over temperature, $\alpha_0 = \mu/T$, and the fluid dynamical flow of matter u^μ . The formula in Eq. (19) reduces to the well known Jüttner [36,37] distribution or relativistic Maxwell-Boltzmann distribution for $q = 1$, that is $f_{Jk} = \exp(\alpha_0 - \beta_0 k^\mu u_\mu)$.

We introduce the q -modified particle four-current and symmetric energy-momentum tensor,

$$N^\mu = \int dK k^\mu \tilde{f}_k, \quad (20)$$

$$T^{\mu\nu} = \int dK k^\mu k^\nu \tilde{f}_k. \quad (21)$$

Now, using Eq. (13) the four-divergences of the above quantities lead to,

$$\partial_\mu N^\mu \equiv \int dK k^\mu \partial_\mu \tilde{f}_k = \int dK C[f], \quad (22)$$

$$\partial_\mu T^{\mu\nu} \equiv \int dK k^\nu k^\mu \partial_\mu \tilde{f}_k = \int dK k^\nu C[f]. \quad (23)$$

Specifying the so-called collision or summation invariants, $\psi = \alpha - \beta_\mu k^\mu$, the conservation of particle four-current and energy-momentum tensor are implied due to the conservation of particle number and energy-momentum in individual collisions. These follow from $\int dK \alpha C[f] = 0$ and $\int dK \beta_\mu k^\mu C[f] = 0$ for any solution of the Boltzmann equation,

$$\partial_\mu N^\mu = 0, \quad (24)$$

$$\partial_\mu T^{\mu\nu} = 0. \quad (25)$$

Having all conserved quantities at hand it is straightforward to prove that the fundamental relation of equilibrium thermodynamics is given as,

$$S^\mu = -\alpha_0 N^\mu + \beta_0 T^{\mu\nu} u_\nu + \beta_0 p_0 u^\mu, \quad (26)$$

where p_0 is the equilibrium pressure. Projecting the above equation with the local flow velocity, we get the familiar form of the thermodynamical relation,

$$s_0 = -\alpha_0 n_0 + \beta_0 (e_0 + p_0). \quad (27)$$

Here $n_0 = \mathcal{J}_{q(1,0)}$ is the particle density, $e_0 = \mathcal{J}_{q(2,0)}$ is the energy density and $p_0 = -\mathcal{J}_{q(2,1)}$ is related to the particle density as $\frac{\partial p_0}{\partial \alpha_0} \Big|_{\beta_0} = \frac{n_0}{\beta_0}$. We introduced the following q -generalized thermodynamic integrals [25]:

$$\mathcal{I}_{q(i,j)} = \frac{1}{(2j+1)!!} \int dK (E_k)^{i-2j} (\Delta^{\mu\nu} k_\mu k_\nu)^j f_{0k}, \quad (28)$$

$$\mathcal{J}_{q(i,j)} = \frac{1}{(2j+1)!!} \int dK (E_k)^{i-2j} (\Delta^{\mu\nu} k_\mu k_\nu)^j (f_{0k})^q, \quad (29)$$

$$\mathcal{K}_{q(i,j)} = \frac{q}{(2j+1)!!} \int dK (E_k)^{i-2j} (\Delta^{\mu\nu} k_\mu k_\nu)^j (f_{0k})^{2q-1}, \quad (30)$$

where $i, j \geq 0$ are natural numbers and the double factorial is $(2j+1)!! = (2j+1)!/(j!2^j)$. Furthermore, we introduced $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$, to project arbitrary four-vectors into other four-vectors orthogonal to u^μ . The particle four-momentum is decomposed as $k^\mu = E_k u^\mu + k^{(\mu)}$, where $E_k = k^\mu u_\mu$ is the Local Rest Frame (LRF) energy of the particle and $k^{(\mu)} = \Delta^{\mu\nu} k_\nu$ the LRF momentum. The LRF is defined by $u_{LRF}^\mu = (1, 0, 0, 0)$.

It is important to realize that the previously introduced thermodynamical integrals are consistent with basic relations of classical thermodynamics. At fixed temperature $(f_{0k})^q = (\partial f_{0k}/\partial \alpha_0)|_{\beta_0}$ as well as $q(f_{0k})^{2q-1} = (\partial (f_{0k})^q/\partial \alpha_0)|_{\beta_0}$, so one obtains

$$\mathcal{J}_{q(i,j)} = \left(\frac{\partial \mathcal{I}_{q(i,j)}}{\partial \alpha_0} \right)_{\beta_0}, \quad \mathcal{K}_{q(i,j)} = \left(\frac{\partial \mathcal{J}_{q(i,j)}}{\partial \alpha_0} \right)_{\beta_0}. \quad (31)$$

One shows by partial integration that the following recursive relations hold in equilibrium:

$$\mathcal{J}_{q(i,j)} = -\frac{1}{\beta_0} \mathcal{I}_{q(i-1,j-1)} + \frac{i-2j}{\beta_0} \mathcal{I}_{q(i-1,j)}, \quad (32)$$

$$\mathcal{K}_{q(i,j)} = -\frac{1}{\beta_0} \mathcal{J}_{q(i-1,j-1)} + \frac{i-2j}{\beta_0} \mathcal{J}_{q(i-1,j)}. \quad (33)$$

Furthermore,

$$\dot{\mathcal{I}}_{q(i,j)} = \mathcal{J}_{q(i,j)} \dot{\alpha}_0 - \mathcal{J}_{q(i+1,j)} \dot{\beta}_0, \quad (34)$$

$$\dot{\mathcal{J}}_{q(i,j)} = \mathcal{K}_{q(i,j)} \dot{\alpha}_0 - \mathcal{K}_{q(i+1,j)} \dot{\beta}_0. \quad (35)$$

These equations provide the q -generalized Gibbs-Duhem relations. They not only reduce to the standard BG thermodynamics for $q = 1$ but at the same time obey relations consistent with standard thermodynamics [25,35]. Similar recursion relations have been obtained in framework of higher order generalized thermodynamics [38,39,40,41].

3 Out of q -equilibrium

So far we have obtained a local q -equilibrium distribution, but yet we are interested in a more general solution of the NEBE. Assume that

$$f_k = f_{0k} + \delta f_k, \quad (36)$$

where δf_k denotes the deviation from local q -equilibrium. It is customary to surmise $\delta f_k \ll f_{0k}$. One should not mix this with an approximation $(q-1) \ll 1$, as pointed out in Refs. [22,23,24]. This distinction becomes physically important when considering the entropy production rate. Namely, the small $(q-1)$ approximation can spoil the positivity of the exact expressions in its leading order. This problem is based on the approximate behaviour of the basic deformed exponential: While

$$e_q(x) = \exp \left\{ \frac{1}{1-q} \ln [1 + (1-q)x] \right\} \geq 0, \quad (37)$$

for any real argument, its small $(q-1)$ approximation,

$$e_q(x) = e^x \cdot \left(1 - (1-q) \frac{x^2}{2} + \mathcal{O}[(q-1)^2] \right), \quad (38)$$

will have a negative sign for $|x| > \sqrt{\frac{2}{1-q}}$. Furthermore, even when $q > 1$ would be postulated, in some formulas $q^* = 2 - q < 1$ occurs - as it will be apparent later. Therefore such approximations loose the positivity property and should not be used in describing dissipative phenomena.

The formula (36) can be approximated by a series expansion around local q -equilibrium. As previously discussed, in local q -equilibrium we may define the collision invariant $\psi_{0k} \equiv \ln_q(f_{0k}) = \alpha_0 - \beta_0 k^\mu u_\mu$, hence

$$\begin{aligned} f_k(\psi) &\equiv f_k(\psi_{0k}) + \frac{\partial f_k(\psi_{0k})}{\partial \psi_{0k}} \phi_k + \mathcal{O}[\phi_k^2] \\ &= f_{0k} + (f_{0k})^q \phi_k + \mathcal{O}[\phi_k^2], \end{aligned} \quad (39)$$

where $\phi_k = (\psi - \psi_{0k})$ denotes the difference between the collision invariants. Therefore comparing the above result with Eq. (36) we have to apply

$$\delta f_k = \tilde{f}_k \phi_k. \quad (40)$$

Because the primary physical quantities are associated with $(f_{0k})^q$ henceforth $\delta \tilde{f}_k$ has to be calculated too. To do this, we expand $\tilde{f}_k = \tilde{f}_{0k} + \delta \tilde{f}_k$ around \tilde{f}_{0k} ,

$$\begin{aligned} \tilde{f}_k &\equiv (f_{0k} + \tilde{f}_{0k} \phi_k + \mathcal{O}[\phi_k^2])^q \\ &= \tilde{f}_{0k} + q(f_{0k})^{2q-1} \phi_k + \mathcal{O}[\phi_k^2], \end{aligned} \quad (41)$$

and so we obtain,

$$\delta \tilde{f}_k = q(f_{0k})^{2q-1} \phi_k. \quad (42)$$

Let us recall the q -generalized Stosszahlansatz (17). To further simplify our discussion we expand it in series,

$$\begin{aligned} H_q[f_k, f_{k'}] &= H_q[f_{0k}, f_{0k'}] + \frac{\partial H_q[f_{0k}, f_{0k'}]}{\partial f_{0k}} \delta f_k \\ &\quad + \frac{\partial H_q[f_{0k}, f_{0k'}]}{\partial f_{0k'}} \delta f_{k'} + \mathcal{O}[\delta f^2], \end{aligned} \quad (43)$$

where

$$\frac{\partial H_q[f_{0k}, f_{0k'}]}{\partial f_{0k}} = (H_q[f_{0k}, f_{0k'}])^q (f_{0k})^{-q}, \quad (44)$$

$$\frac{\partial H_q[f_{0k}, f_{0k'}]}{\partial f_{0k'}} = (H_q[f_{0k}, f_{0k'}])^q (f_{0k'})^{-q}. \quad (45)$$

Now, making use of Eq. (40) we arrive at,

$$H_q[f_k, f_{k'}] = H_q[f_{0k}, f_{0k'}] + (H_q[f_{0k}, f_{0k'}])^q (\phi_k + \phi_{k'}). \quad (46)$$

In this way the collision integral from Eq. (16) is approximated up to first order in deviations from q -equilibrium as,

$$C[f] = C[f_0] + C[\delta f]. \quad (47)$$

Here $C[f_0] \propto H_q[f_{0p}, f_{0p'}] - H_q[f_{0k}, f_{0k'}]$ vanishes by definition, $H_q[f_{0p}, f_{0p'}] = H_q[f_{0k}, f_{0k'}]$. Then $C[f] = C[\delta f]$ is given by the integral,

$$C[\delta f] = \frac{1}{2} \int dK' dP dP' W_{kk'l \rightarrow pp'} (H_q[f_{0k}, f_{0k'}])^q \times (\phi_p + \phi_{p'} - \phi_k - \phi_{k'}), \quad (48)$$

and so the NEBE (13) simplifies to, $k^\mu \partial_\mu \tilde{f}_k = C[\delta f]$.

Using the out-of- q -equilibrium distribution function, the particle four-current and the energy-momentum tensor from Eqs. (20,21) take the general form,

$$N^\mu = n u^\mu + V^\mu, \quad (49)$$

$$T^{\mu\nu} = e u^\mu u^\nu - p \Delta^{\mu\nu} + u^\mu W^\nu + u^\nu W^\mu + \pi^{\mu\nu}. \quad (50)$$

We can uniquely identify the fundamental fluid dynamical quantities like the particle density, energy density and isotropic pressure by following the matching procedure,

$$n \equiv u_\mu N^\mu = \int dK E_k \tilde{f}_k, \quad (51)$$

$$e \equiv u_\mu u_\nu T^{\mu\nu} = \int dK E_k^2 \tilde{f}_k, \quad (52)$$

$$p \equiv -\frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu} = -\frac{1}{3} \int dK (\Delta^{\alpha\beta} k_\alpha k_\beta) \tilde{f}_k. \quad (53)$$

Similarly the particle diffusion and energy-momentum four-currents as well as the stress tensor are defined as,

$$V^\mu \equiv \Delta_\alpha^\mu N^\alpha = \int dK k^{\langle\mu} \delta \tilde{f}_k, \quad (54)$$

$$W^\mu \equiv \Delta_\alpha^\mu u_\beta T^{\alpha\beta} = \int dK E_k k^{\langle\mu} \delta \tilde{f}_k, \quad (55)$$

$$\pi^{\mu\nu} \equiv T^{\langle\mu\nu\rangle} = \int dK k^{\langle\mu} k^{\nu\rangle} \delta \tilde{f}_k, \quad (56)$$

where $\delta \tilde{f}_k = \tilde{f}_k - \tilde{f}_{0k}$ and we used the following constraint $\int dK (E_k)^l (\Delta^{\mu\nu} k_\mu k_\nu)^m k^{\langle\mu_1} \dots k^{\mu_n\rangle} \tilde{f}_{0k} = 0$ for any l, n, m natural numbers. Here $T^{\langle\mu\nu\rangle} = \Delta^{\mu\nu\alpha\beta} T_{\alpha\beta}$, with $\Delta^{\mu\nu\alpha\beta} = \frac{1}{2} (\Delta^{\mu\alpha} \Delta^{\beta\nu} + \Delta^{\nu\alpha} \Delta^{\beta\mu}) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\alpha\beta}$ projector, selecting out the traceless, symmetric and orthogonal to u_μ part of $T^{\mu\nu}$.

The particle four-current and the energy-momentum tensor calculated from Eqs. (20,21) by substituting the local q -equilibrium distribution function from Eq. (19), leads to the so-called *perfect* q -fluid decomposition,

$$N_0^\mu = n_0 u^\mu, \quad (57)$$

$$T_0^{\mu\nu} = e_0 u^\mu u^\nu - p_0 \Delta^{\mu\nu}, \quad (58)$$

where $n_0 \equiv u_\mu N_0^\mu = \int dK E_k \tilde{f}_{0k}$ is the particle density, $e_0 \equiv u_\mu u_\nu T_0^{\mu\nu} = \int dK E_k^2 \tilde{f}_{0k}$ is the energy density and $p_0 \equiv -\frac{1}{3} \Delta_{\mu\nu} T_0^{\mu\nu} = -\frac{1}{3} \int dK (\Delta^{\alpha\beta} k_\alpha k_\beta) \tilde{f}_{0k}$ is the pressure in q -equilibrium. The particle density and energy density are usually assumed to be unchanged from their equilibrium values $n = n_0$ and $e = e_0$, also cited as the Landau

matching conditions,

$$\delta n \equiv \int dK E_k \delta \tilde{f}_k = 0, \quad (59)$$

$$\delta e \equiv \int dK E_k^2 \delta \tilde{f}_k = 0. \quad (60)$$

The isotropic pressure p separates into two parts: $p = p_0 + \Pi$ where the bulk viscous pressure becomes,

$$\Pi = -\frac{1}{3} \int dK (\Delta^{\alpha\beta} k_\alpha k_\beta) \delta \tilde{f}_k. \quad (61)$$

The conservation equations for N_0^μ and $T_0^{\mu\nu}$ turn to the Euler equations for a perfect q -fluid, while Eqs. (24,25) together with Eqs. (49, 50) constitute the equations of dissipative q -fluid dynamics.

Although we motivated the q -fluid dynamical equations from the NEBE, these constituting equations can also be *postulated* as the q -generalized version of the conservation equations based on the non-extensive q -entropy four-current and the corresponding laws of thermodynamics. The equations of perfect q -fluid dynamics are closed by an Equation of State (EoS). Without a first order phase transition it implies a vanishing local entropy production in q -equilibrium, $\partial_\mu S_0^\mu = 0$. However, the equations of q -fluid dynamics presented so far in this paper are not closed because the out-of- q -equilibrium \tilde{f}_k is unknown. Therefore in this case one needs additional relations beyond the EoS for completing the system of equations.

4 The Navier-Stokes equations and transport coefficients

Taking advantage of the previously derived approximations we reduce the problem to finding $\delta f_k = \tilde{f}_{0k} \phi_k$ near local q -equilibrium. There are several well known approximations to obtain the deviations from local equilibrium as well as to calculate the transport coefficients. Here we follow the traditional approach of Chapman and Enskog [40,41,42] and apply this method for the NEBE. To this end we note that this method is iterative, and here we take only the first step, which results in the well known Navier-Stokes equations of fluid dynamics.

Assuming that in the vicinity of local q -equilibrium $\delta \tilde{f}_k \ll \tilde{f}_{0k}$, the non-equilibrium contributions from the streaming term are also assumed to vanish $k^\mu \partial_\mu \delta \tilde{f}_k = 0$. Therefore $k^\mu \partial_\mu \tilde{f}_k \simeq k^\mu \partial_\mu \tilde{f}_{0k}$, i.e.,

$$k^\mu \partial_\mu \tilde{f}_{0k} = q f_{0k}^{2q-1} \left[-\beta_0 \frac{\theta}{3} (k^\alpha k^\beta \Delta_{\alpha\beta}) + E_k \dot{\alpha}_0 - E_k^2 \dot{\beta}_0 + (1 - E_k h_0^{-1}) k^{\langle\mu} \nabla_\mu \alpha_0 - \beta_0 k^{\langle\mu} k^{\nu\rangle} \sigma_{\mu\nu} \right]. \quad (62)$$

Here, $\theta = \nabla_\mu u^\mu$ is the expansion scalar, $\sigma^{\mu\nu} = \nabla^{\langle\mu} u^{\nu\rangle}$ is the shear-stress tensor and $\nabla_\mu \alpha_0$ is the gradient of

the chemical potential over temperature. The proper time derivatives $\dot{\alpha}_0$ and $\dot{\beta}_0$ can be expressed as

$$\dot{\alpha}_0 = n_0 \mathcal{D}_{q(2,0)}^{-1} (h_0 \mathcal{K}_{q(2,0)} - \mathcal{K}_{q(3,0)}) \theta, \quad (63)$$

$$\dot{\beta}_0 = n_0 \mathcal{D}_{q(2,0)}^{-1} (h_0 \mathcal{K}_{q(1,0)} - \mathcal{K}_{q(2,0)}) \theta, \quad (64)$$

where $h_0 \equiv (e_0 + p_0)/n_0 = \mathcal{K}_{q(3,1)}/\mathcal{K}_{q(2,1)}$ is the enthalpy per particle and $\mathcal{D}_{q(i,j)} = \mathcal{K}_{q(i-1,j)} \mathcal{K}_{q(i+1,j)} - \mathcal{K}_{q(i,j)}^2$.

The main equation we need to solve in the first order Chapman-Enskog method reduces to,

$$k^\mu \partial_\mu \tilde{f}_{0k} = C [\delta f], \quad (65)$$

where the left-hand-side is given by Eq. (62), while $C [\delta f]$ is given by Eq. (48). Inspecting this integro-differential equation, we search for its solution using the ansatz [41],

$$\begin{aligned} \phi_k = & (\varphi_0 + \varphi_1 E_{\mathbf{k}} + \varphi_2 E_{\mathbf{k}}^2) \theta + (\varphi_3 + \varphi_4 E_{\mathbf{k}}) k^{(\mu)} \nabla_\mu \alpha_0 \\ & + \varphi_5 k^{(\mu} k^{\nu)} \sigma_{\mu\nu}, \end{aligned} \quad (66)$$

where the φ_i 's are yet unknown coefficients. In an earlier work [25] we calculated the transport coefficients based on the Anderson-Witting model [43] where ϕ_k was given directly by Eq. (62) using the so-called relaxation time approximation, $C [\delta f] = -E_{\mathbf{k}} (\tilde{f}_k - \tilde{f}_{0k}) / \tau_C$, with τ_C being the mean time between collisions. Similar calculations were also done in the non-relativistic limit by Bezerra et al. [44] using an alternative NEBE from Refs. [29,30].

Here we proceed with the calculation of the φ_i coefficients. Using Eqs. (59,60) and the definition of the bulk viscous pressure from Eq. (61) together with Eq. (42) we get three coupled equations for the first three coefficients:

$$\begin{aligned} \delta n \equiv & q \int dK E_{\mathbf{k}} (f_{0k})^{2q-1} \phi_k \\ = & (\varphi_0 \mathcal{K}_{q(1,0)} + \varphi_1 \mathcal{K}_{q(2,0)} + \varphi_2 \mathcal{K}_{q(3,0)}) \theta, \end{aligned} \quad (67)$$

$$\begin{aligned} \delta e \equiv & q \int dK E_{\mathbf{k}}^2 (f_{0k})^{2q-1} \phi_k \\ = & (\varphi_0 \mathcal{K}_{q(2,0)} + \varphi_1 \mathcal{K}_{q(3,0)} + \varphi_2 \mathcal{K}_{q(4,0)}) \theta, \end{aligned} \quad (68)$$

$$\begin{aligned} \Pi \equiv & -\frac{q}{3} \int dK (\Delta^{\alpha\beta} k_\alpha k_\beta) (f_{0k})^{2q-1} \phi_k \\ = & -(\varphi_0 \mathcal{K}_{q(2,1)} + \varphi_1 \mathcal{K}_{q(3,1)} + \varphi_2 \mathcal{K}_{q(4,1)}) \theta. \end{aligned} \quad (69)$$

For $\delta n = \delta e = 0$ from the matching conditions (59,60), one concludes that

$$\varphi_0 = -\frac{\Pi}{\theta} \frac{\mathcal{D}_{q(3,0)}}{\mathcal{D}_\Pi}, \quad (70)$$

$$\varphi_1 = -\frac{\Pi}{\theta} \frac{(\mathcal{K}_{q(2,0)} \mathcal{K}_{q(3,0)} - \mathcal{K}_{q(1,0)} \mathcal{K}_{q(4,0)})}{\mathcal{D}_\Pi}, \quad (71)$$

$$\varphi_2 = -\frac{\Pi}{\theta} \frac{\mathcal{D}_{q(2,0)}}{\mathcal{D}_\Pi}, \quad (72)$$

where the denominator is given by

$$\begin{aligned} \mathcal{D}_\Pi = & \mathcal{K}_{q(2,1)} \mathcal{D}_{q(3,0)} + \mathcal{K}_{q(4,1)} \mathcal{D}_{q(2,0)} \\ & + \mathcal{K}_{q(3,1)} (\mathcal{K}_{q(2,0)} \mathcal{K}_{q(3,0)} - \mathcal{K}_{q(1,0)} \mathcal{K}_{q(4,0)}). \end{aligned} \quad (73)$$

It is important to note that the matching conditions are not unique and other choices may be made [45,46,47]. In general one may assume that $\delta n \neq \delta e \neq 0$, so the coefficients $\varphi_i (\delta n, \delta e, \Pi)$ are functions of all scalar non-equilibrium corrections. However, as we shall see it later, the constraints to the NEBE only supplies φ_2 , the other two coefficients are left undetermined. Therefore, in general we need additional input to specify the non-equilibrium scalar corrections, and in this paper we work with the matching conditions from Eqs. (59,60).

Similarly, the definitions of the particle and energy-momentum diffusion four-currents and the stress tensor lead to,

$$\begin{aligned} V^\mu \equiv & q \int dK k^{(\mu)} (f_{0k})^{2q-1} \phi_k \\ = & (\varphi_3 \mathcal{K}_{q(2,1)} + \varphi_4 \mathcal{K}_{q(3,1)}) \nabla^\mu \alpha_0, \end{aligned} \quad (74)$$

$$\begin{aligned} W^\mu \equiv & q \int dK E_{\mathbf{k}} k^{(\mu)} (f_{0k})^{2q-1} \phi_k \\ = & (\varphi_3 \mathcal{K}_{q(3,1)} + \varphi_4 \mathcal{K}_{q(4,1)}) \nabla^\mu \alpha_0, \end{aligned} \quad (75)$$

$$\begin{aligned} \pi^{\mu\nu} \equiv & q \int dK k^{(\mu} k^{\nu)} (f_{0k})^{2q-1} \phi_k \\ = & 2\varphi_5 \mathcal{K}_{q(4,2)} \sigma^{\mu\nu}. \end{aligned} \quad (76)$$

Here we used, $k^{(\mu} k^{\nu)} = k^{(\mu} k^{(\nu)} - \frac{1}{3} (\Delta^{\alpha\beta} k_\alpha k_\beta) \Delta^{\mu\nu}$, and the general orthogonality relation,

$$\begin{aligned} & \int dK F(E_{\mathbf{k}}) k^{(\mu_1} \dots k^{\mu_m)} k_{(\nu_1} \dots k_{\nu_n)} \\ = & \frac{m! \delta_{mn}}{(2m+1)!!} \Delta_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_m} \int dK F(E_{\mathbf{k}}) (\Delta^{\alpha\beta} k_\alpha k_\beta)^m, \end{aligned} \quad (77)$$

with $F(E_{\mathbf{k}})$ being a function of energy [40].

These results for the dissipative quantities present the relativistic Navier-Stokes relations, usually written in the more familiar form,

$$\Pi = -\zeta \theta, \quad (78)$$

$$q^\mu = -\frac{\kappa}{h_0 \beta_0^2} \nabla^\mu \alpha_0, \quad (79)$$

$$\pi^{\mu\nu} = 2\eta \sigma^{\mu\nu}. \quad (80)$$

The relativistic counterpart of the Stokes result, $\Pi = -\zeta \theta$, relates the bulk viscous pressure to the expansion rate by introducing the coefficient of bulk viscosity ζ . The Fourier-Navier-Stokes law in Eq. (79) relates the heat-flow $q^\mu = W^\mu - h_0 V^\mu$ to the temperature gradients in the system, $\nabla^\mu \alpha_0 = -h_0 T^{-2} (\nabla^\mu T - T \dot{u}^\mu)$ with the coefficient of heat- or thermal-conductivity κ . Finally the Newton-Navier-Stokes relation between the stress and shear is given in Eq. (80) with the coefficient of shear viscosity η .

The φ_i coefficients are obtained inserting ϕ_k from Eq. (66) into the collision integral (48) and matching the gra-

dients on both sides. The collision integral leads to,

$$\begin{aligned} C[\delta f] = & \frac{1}{2} \int dK' dP dP' W_{kk' \rightarrow pp'} (H_q[f_{0k}, f_{0k'}])^q \\ & \times [\varphi_2 (E_p^2 + E_{p'}^2 - E_k^2 - E_{k'}^2) \theta \\ & + \varphi_4 (E_p p^{(\mu)} + E_{p'} p'^{(\mu)} - E_k k^{(\mu)} - E_{k'} k'^{(\mu)}) \nabla_\mu \alpha_0 \\ & + \varphi_5 (p^{(\mu} p^{\nu)} + p'^{(\mu} p'^{\nu)} - k^{(\mu} k^{\nu)} - k'^{(\mu} k'^{\nu)}) \sigma_{\mu\nu}] , \end{aligned} \quad (81)$$

where the contributions vanish identically for φ_0 , φ_1 and φ_3 due to the conservation of charge, energy and momenta in binary collisions.

For later purposes we introduce the $X^{\mu\nu\alpha\beta}$ tensor, symmetric upon the interchange of indexes (μ, ν) as well as (α, β) , while traceless for the latter, $X^{\mu\nu\alpha\beta} g_{\alpha\beta} = 0$:

$$\begin{aligned} X^{\mu\nu\alpha\beta} = & \frac{1}{2} \int dK dK' dP dP' W_{kk' \rightarrow pp'} (H_q[f_{0k}, f_{0k'}])^q \\ & \times k^\mu k^\nu (p^\alpha p^\beta + p'^\alpha p'^\beta - k^\alpha k^\beta - k'^\alpha k'^\beta) . \end{aligned} \quad (82)$$

Furthermore, one can show that $X^{\mu\nu\alpha\beta}$ is generally decomposed as,

$$\begin{aligned} X^{\mu\nu\alpha\beta} = & (X_1 u^\mu u^\nu + X_2 \Delta^{\mu\nu}) \left(u^\alpha u^\beta - \frac{1}{3} \Delta^{\alpha\beta} \right) \\ & + 4X_3 u^{(\mu} \Delta^{\nu)(\alpha} u^{\beta)} + X_4 \Delta^{\mu\nu\alpha\beta} , \end{aligned} \quad (83)$$

with the coefficients,

$$X_1 \equiv X^{\mu\nu\alpha\beta} u_\mu u_\nu u_\alpha u_\beta = -X^{\mu\nu\alpha\beta} u_\mu u_\nu \Delta_{\alpha\beta} , \quad (84)$$

$$X_2 \equiv \frac{1}{3} X^{\mu\nu\alpha\beta} \Delta_{\mu\nu} u_\alpha u_\beta = -\frac{1}{3} X^{\mu\nu\alpha\beta} \Delta_{\mu\nu} \Delta_{\alpha\beta} , \quad (85)$$

$$X_3 \equiv \frac{1}{3} X^{\mu\nu\alpha\beta} u_{(\mu} \Delta_{\nu)(\alpha} u_{\beta)} = \frac{1}{3} X^{\mu\nu\alpha\beta} u_\mu \Delta_{\nu\alpha} u_\beta , \quad (86)$$

$$X_4 \equiv \frac{1}{5} X^{\mu\nu\alpha\beta} \Delta_{\mu\nu\alpha\beta} . \quad (87)$$

Applying these results and integrating Eq. (65) we get three equations which are proportional to the different Navier-Stokes type gradients occurring in Eqs. (78-80). For example, collecting terms proportional to $\sigma^{\alpha\beta}$ leads to,

$$\begin{aligned} & -\beta_0 \int dK k^\mu k^\nu k_{(\alpha} k_{\beta)} (q f_{0k}^{2q-1}) \\ = & \frac{\varphi_5}{2} \int dK dK' dP dP' W_{kk' \rightarrow pp'} (H_q[f_{0k}, f_{0k'}])^q \\ & \times k^\mu k^\nu (p_{(\alpha} p_{\beta)} + p'_{(\alpha} p'_{\beta)} - k_{(\alpha} k_{\beta)} - k'_{(\alpha} k'_{\beta)}) , \end{aligned} \quad (88)$$

which briefly reads as, $-2\beta_0 \mathcal{K}_{q(4,2)} \Delta^{\mu\nu\alpha\beta} = \varphi_5 X^{\mu\nu\alpha\beta}$. From this we easily get that

$$\varphi_5 = -2\beta_0 \frac{\mathcal{K}_{q(4,2)}}{X_4} . \quad (89)$$

Replacing this result into Eq. (76) and comparing it to Eq. (80) leads to the coefficient of shear viscosity,

$$\eta \equiv \varphi_5 \mathcal{K}_{q(4,2)} = -2\beta_0 \frac{\mathcal{K}_{q(4,2)}^2}{X_4} . \quad (90)$$

Similarly, the coefficient of heat-conductivity can be calculated, by matching the gradient $\nabla^\mu \alpha_0$ on both sides of the integrated NEBE. This leads to,

$$\varphi_4 = -\frac{\mathcal{K}_{q(3,1)} - h_0^{-1} \mathcal{K}_{q(4,1)}}{X_3} . \quad (91)$$

The φ_3 coefficient can be found only after fixing the LR frame. Here we have at least two choices. We may choose $V^\mu = 0$, which corresponds to Eckart's definition [48], and hence the fluid dynamical flow velocity is fixed to the flow of conserved particles, $u^\mu = N^\mu / \sqrt{N^\mu N_\mu}$. Another choice due to Landau and Lifshitz [49] defines the flow as the time-like eigenvector of the energy-momentum tensor, $u^\mu = T^{\mu\nu} u_\nu / \sqrt{T^{\mu\alpha} u_\alpha T_{\mu\beta} u^\beta}$. This choice is equivalent to $W^\mu = 0$. Choosing the Landau frame, one arrives at

$$\varphi_3 = -\varphi_4 \frac{\mathcal{K}_{q(4,1)}}{\mathcal{K}_{q(3,1)}} , \quad (92)$$

and thus the coefficient of heat conductivity becomes

$$\kappa = \varphi_4 h_0^2 \beta_0^2 \frac{\mathcal{D}_{q(3,1)}}{\mathcal{K}_{q(3,1)}} . \quad (93)$$

Finally, the coefficient of bulk viscosity can be expressed with the help of Eqs. (63, 64):

$$\begin{aligned} \varphi_2 = & -\frac{\mathcal{D}_{q(2,0)}^{-1}}{X_1} [\beta_0 \mathcal{K}_{q(4,1)} \mathcal{D}_{q(2,0)} - n_0 \mathcal{D}_{q(3,0)} \\ & - n_0 h_0 (\mathcal{K}_{q(2,0)} \mathcal{K}_{q(3,0)} - \mathcal{K}_{q(1,0)} \mathcal{K}_{q(4,0)})] , \end{aligned} \quad (94)$$

and hence from Eqs. (72) and (78) we obtain,

$$\zeta = \varphi_2 \frac{\mathcal{D}_\Pi}{\mathcal{D}_{q(2,0)}} . \quad (95)$$

These results formally resemble their classical counterparts within BG statistics. An explicit q -dependence occurs in the thermodynamic integrals only while the coefficients X_1, X_2, X_3 and X_4 differ in their arguments.

In the followings we attempt to simplify the collisional integral. Recall the $H_q[f_k, f_{k'}]$ functional from Eq. (17) and calculate it in equilibrium getting

$$\begin{aligned} (H_q[f_{0k}, f_{0k'}])^q & \equiv (\exp_q[\ln_q(f_{0k}) + \ln_q(f_{0k'})])^q \\ & = [1 + (1 - q)(\psi_{0k} + \psi_{0k'})]^{\frac{q}{1-q}} . \end{aligned} \quad (96)$$

Now making use of Eq. (3) one concludes that

$$H_q[f_{0k}, f_{0k'}] = \exp_q(\psi_{0k} + \psi_{0k'}) . \quad (97)$$

The above formula for $q = 1$ leads to the classical result, $H_1[f_{0k}, f_{0k'}] = f_{Jk} f_{Jk'}$ with $f_{Jk} \equiv \exp(\psi_{0k}) = \exp(\alpha_0 - \beta_0 k^\mu u_\mu)$. Substituting this result into Eq. (82) we arrive at,

$$\begin{aligned} X^{\mu\nu\alpha\beta} = & \frac{1}{2} \int dK dK' dP dP' W_{kk' \rightarrow pp'} (\exp_q(\psi_{0k} + \psi_{0k'}))^q \\ & \times k^\mu k^\nu (p^\alpha p^\beta + p'^\alpha p'^\beta - k^\alpha k^\beta - k'^\alpha k'^\beta) . \end{aligned} \quad (98)$$

This integral can be rewritten with the help of a Mandelstam invariant $s \equiv (k^\mu + k'^\mu)^2 = (p^\mu + p'^\mu)^2$. So the transition rate is given as [40],

$$W_{kk' \rightarrow pp'} = (2\pi)^6 s \sigma(s, \theta_{CM}) \delta^4(k^\mu + k'^\mu - p^\mu - p'^\mu). \quad (99)$$

Here $\sigma(s, \theta_{CM})$ is the differential cross section, θ_{CM} is the scattering angle in the center of momentum frame, while the δ -function represents the energy-momentum conservation in binary collisions.

Substituting the transition rate into Eq. (98), we get the following expression in the center of mass frame for the p -dependent integral:

$$\begin{aligned} \mathcal{I}_p(\sigma_T, s, m) &\equiv \frac{1}{2} \int \frac{d^3\mathbf{p}}{(p^0)^2} s \sigma(s, \theta_{CM}) \delta(\sqrt{s} - 2p^0), \\ &= \frac{\sigma_T(s)}{2} \sqrt{s(s - 4m^2)}. \end{aligned} \quad (100)$$

Here we defined the total cross-section $\sigma_T(s)$ as the integral over the differential cross-section and the solid angle, $d\Omega = \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta_{CM} d\theta_{CM}$, hence

$$\sigma_T(s) = \frac{1}{2} \int_0^\pi d\Omega \sigma(s, \theta_{CM}). \quad (101)$$

Finally the $X^{\mu\nu\alpha\beta}$ tensor can be given as,

$$\begin{aligned} X^{\mu\nu\alpha\beta} &= \frac{1}{2} \int dK dK' \mathcal{I}_p(\sigma_T, s, m) (\exp_q(\psi_{0k} + \psi_{0k'}))^q \\ &\times k^\mu k^\nu \left\{ \frac{1}{2} \left[P^\alpha P^\beta - \frac{(s - 4m^2)}{3} \left(g^{\alpha\beta} - \frac{P^\alpha P^\beta}{s} \right) \right] \right. \\ &\left. - (k^\alpha k^\beta + k'^\alpha k'^\beta) \right\}, \end{aligned} \quad (102)$$

where we introduced the total momentum $P^\mu \equiv k^\mu + k'^\mu = p^\mu + p'^\mu$. This integral, contrary to the classical examples is not easy to handle analytically due to the fact that only $\exp_q[x + y + (1 - q)xy] = \exp_q(x) \exp_q(y)$, hence the k integrals can not be factorized into thermodynamical integrals.

As already shown the thermodynamical quantities and transport coefficients change with q -parameter. The fact that the collision integral does not factorize, signals a direct q -dependence and the effect of the correlations included in the NEBE. Therefore it also follows that the original Boltzmann equation ($q = 1$) neglects any kind of memory effects and two particle correlations, hence the transport coefficients calculated from such an equation also lack correlations.

5 Conclusions and outlook

In this work we derived the q -generalized versions of the classical Navier-Stokes equations of relativistic dissipative fluid dynamics from a q -generalized Boltzmann transport equation. These equations were found based on the Chapman-Enskog method.

We showed that starting from a q -generalized transport equation, it is justified to apply standard methods to calculate transport coefficients. These calculations lead to relations for all transport coefficients formally similar to those one would obtain using the traditional Boltzmann-Gibbs distributions. The main difference is contained in the recursive rules for the thermodynamic integrals.

However, unlike in traditional fluid dynamics, the tensorial collision kernel $X^{\mu\nu\alpha\beta}$ does not factorize into product of simple thermodynamical integrals. This remarkable property is a consequence of the q -deformed exponential function describing the extended local q -equilibrium distribution.

We also add that the derivation presented in this work is applicable to slightly different kinetic equations as discussed in Appendix A.1, as well as to the one suggested by G. Kaniadakis [3, 50, 51]. Furthermore, we also show in Appendix A.2 that one obtains causal fluid dynamics from the NEBE by using Grad's method moments.

These methods applied to the q -generalized Boltzmann equation extend the applicability of dissipative fluid dynamics for $q \neq 1$ by including long range interactions and correlations, but for example the resulting Navier-Stokes equations are still parabolic, hence problems related to acausality are not solved by introducing a non-extensivity parameter q in this framework. Therefore this means that the causality problem is not rooted in the non-extensivity of entropy, but the entropy should be extended to include dissipative quantities which is well known from irreversible thermodynamic theories [52, 53].

On the other hand, we remark that Osada and Wilk in Refs. [22, 23, 24] associated the perfect q -hydrodynamics with the classical Navier-Stokes equations using a so-called non-extensive/dissipative correspondence (NexDC). This approach makes a direct correspondence between the purely q -dependent conserved quantities, Eqs. (57, 58) for $q \neq 1$, and the general dissipative structure as given in Eqs. (49, 50) for $q = 1$. Therefore the dissipative quantities are identified with differences given by perfect q -fluid dynamical quantities for $q \neq 1$ and classical dissipative fluid dynamics for $q = 1$.

This novel method may reveal the correlations induced by long range effects that in general contribute to dissipation and entropy production. However our approach goes beyond by generalizing the fluid dynamical equations to out of q -equilibrium states, hence in principle one can directly estimate the q -parameter without conjecturing a NexDC.

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A Appendix

A.1 Remarks on a different NEBE

Here we briefly discuss different NEBEs advocated by some authors. The existence of different kinetic equations shows that there is an ambiguity about the correct equation of motion, although they are consistent with the same q -generalized entropy.

Let us recall the entropy four-current from Eq. (10) and rewrite it in equivalent forms:

$$\begin{aligned} S^\mu &\equiv - \int dK \, k^\mu [f_k \ln_{q^*}(f_k) - f_k], \\ &= - \frac{1}{q} \int dK \, k^\mu [f_k \text{Ln}_q(f_k) - q f_k]. \end{aligned} \quad (103)$$

The requirement of positive entropy production turns out to be,

$$\partial_\mu S^\mu \equiv -q \int dK \ln_{q^*}(f_k) [k^\mu \partial_\mu f_k] \geq 0, \quad (104)$$

and

$$\partial_\mu S^\mu = - \int dK \text{Ln}_q(f_k) [k^\mu \partial_\mu f_k] \geq 0. \quad (105)$$

The above formulas suggest a different version of the q -generalized Boltzmann equation. This alternative NEBE can be written as, (denoted with hats to avoid confusion)

$$k^\mu \partial_\mu \hat{f}_k = \hat{C}[\hat{f}]. \quad (106)$$

Here the collision integral is

$$\begin{aligned} \hat{C}[\hat{f}] &= \frac{1}{2} \int dK' dP dP' W_{kk' \rightarrow pp'} \\ &\times \left(\hat{H}_q[\hat{f}_p, \hat{f}_{p'}] - \hat{H}_q[\hat{f}_k, \hat{f}_{k'}] \right), \end{aligned} \quad (107)$$

and

$$\hat{H}_q[\hat{f}_k, \hat{f}_{k'}] = \exp_q \left[\text{Ln}_q(\hat{f}_k) + \text{Ln}_q(\hat{f}_{k'}) \right]. \quad (108)$$

The outcome of the H-theorem as shown by Abe [27], is equivalent to $\partial_\mu S^\mu \equiv - \int dK \text{Ln}_q(f_{0k}) \hat{C}[f] = 0$, whence the collision invariant is $\hat{\psi}_{0k} = \text{Ln}_q(\hat{f}_{0k})$, leading to

$$\hat{f}_{0k} \equiv E_q(\alpha_0 - \beta_0 k^\mu u_\mu) = \exp_{q^*} \left(\frac{\alpha_0 - \beta_0 k^\mu u_\mu}{q} \right). \quad (109)$$

We further note that Lima and Silva et al. [29,30] used a slightly different q -generalized Stosszahlansatz given as $H_q^*[f_k^*, f_{k'}^*] = \exp_q [\ln_{q^*}(f_k^*) + \ln_{q^*}(f_{k'}^*)]$. Actually, this can be achieved by the duality transformation, $\ln_{q^*}(x) = \frac{1}{q} \text{Ln}_q(x)$, and so their equilibrium distribution function becomes,

$$f_{0k}^* = e_{q^*}(\alpha_0 - \beta_0 k^\mu u_\mu). \quad (110)$$

It is clear that the different NEBE's lead to different stationary solutions. However the cited solution of Abe as well as of Lima and Silva et al., consequently leads to different thermodynamical relations than ours.

All these different q -equilibrium distributions can be used to define different particle four-currents and symmetric energy-momentum tensors. For example, denote $N^\mu(\hat{f}_{0k}) = \hat{N}_0^\mu$, $T^{\mu\nu}(\hat{f}_{0k}) = \hat{T}_0^{\mu\nu}$ and $S^\mu(\hat{f}_{0k}) = \hat{S}_0^\mu$, whence by using Eq. (103) together with the corresponding stationary solution (109), we obtain

$$\begin{aligned} \hat{S}_0^\mu &\equiv - \frac{\alpha_0}{q} \int dK k^\mu \hat{f}_{0k} + \frac{\beta_0}{q} \int dK k^\mu E_k \hat{f}_{0k} + \int dK k^\mu \hat{f}_{0k} \\ &= - \frac{\alpha_0}{q} \hat{N}_0^\mu + \frac{\beta_0}{q} \hat{T}_0^{\mu\nu} u_\nu + \hat{N}_0^\mu. \end{aligned} \quad (111)$$

Projecting the above equation and introducing $\hat{\alpha}_0 = \alpha_0/q$ and $\hat{\beta}_0 = \beta_0/q$ we get, $\hat{s}_0 = -\hat{\alpha}_0 \hat{n}_0 + \hat{\beta}_0 \hat{e}_0 + \hat{n}_0$. Correspondingly one has to define, the following thermodynamic integrals, similar to Eqs. (28,29,30),

$$\hat{I}_{q(i,j)} = \frac{1}{(2j+1)!!} \int dK (E_k)^{i-2j} (\Delta^{\mu\nu} k_\mu k_\nu)^j (\hat{f}_{0k})^q, \quad (112)$$

$$\hat{J}_{q(i,j)} = \frac{1}{(2j+1)!!} \int dK (E_k)^{i-2j} (\Delta^{\mu\nu} k_\mu k_\nu)^j \hat{f}_{0k}, \quad (113)$$

$$\hat{K}_{q(i,j)} = \frac{q^{-1}}{(2j+1)!!} \int dK (E_k)^{i-2j} (\Delta^{\mu\nu} k_\mu k_\nu)^j (\hat{f}_{0k})^{2-q}. \quad (114)$$

Now we rewrite the thermodynamic relation as,

$$\hat{s}_0 = -\hat{\alpha}_0 \hat{J}_{q(1,0)} + \hat{\beta}_0 \hat{J}_{q(2,0)} + \hat{J}_{q(1,0)}. \quad (115)$$

However at fixed temperature $\hat{f}_{0k} = \left(\partial (\hat{f}_{0k})^q / \partial \alpha_0 \right) |_{\beta_0}$ and $q^{-1} (\hat{f}_{0k})^{2-q} = \left(\partial \hat{f}_{0k} / \partial \alpha_0 \right) |_{\beta_0}$ hence

$$\hat{J}_{q(i,j)} = \left(\frac{\partial \hat{I}_{q(i,j)}}{\partial \alpha_0} \right)_{\beta_0}, \quad \hat{K}_{q(i,j)} = \left(\frac{\partial \hat{J}_{q(i,j)}}{\partial \alpha_0} \right)_{\beta_0}, \quad (116)$$

while

$$\hat{J}_{q(i,j)} = - \frac{1}{\beta_0} \hat{I}_{q(i-1,j-1)} + \frac{i-2j}{\beta_0} \hat{I}_{q(i-1,j)}. \quad (117)$$

These relations actually lead to an equilibrium pressure given as, $\hat{p}_0 \equiv -\hat{J}_{q(2,1)} = \beta_0^{-1} \hat{J}_{q(1,0)}$ and so the last term from Eq. (115) cannot be identified with the pressure times inverse temperature $\hat{\beta}_0 \hat{p}_0 \neq \hat{J}_{q(1,0)}$. Therefore the equilibrium state defined by Eq. (109) and Eq. (110), are not consistent with the ideal gas EOS in its classical form, while the one utilized in Eq. (19) is, $\beta_0 p_0 = \mathcal{I}_{q(1,0)}$. This does not necessarily mean that the NEBE in Eq. (106) or its stationary solutions are ruled out, but one has to keep in mind that they lead to weird thermodynamic relations.

Similarly, as previously presented we can define the deviations from equilibrium as,

$$\hat{f}_k = \hat{f}_{0k} + \frac{1}{q} \left(\hat{f}_{0k} \right)^{2-q} \hat{\phi}_k + \mathcal{O} \left[\hat{\phi}_k^2 \right],$$

where $\hat{\phi}_k = (\hat{\psi} - \hat{\psi}_{0k})$ and so $\delta \hat{f}_k = \frac{1}{q} \left(\hat{f}_{0k} \right)^{2-q} \hat{\phi}_k$. The collision integral also simplifies and formally corresponds to Eq. (48),

$$\begin{aligned} \hat{C}[\delta f] &= \frac{1}{2} \int dK' dP dP' W_{kk't \rightarrow pp'} \left(\hat{H}_q \left[\hat{f}_{0k}, \hat{f}_{0k'} \right] \right)^q \\ &\times \left(\hat{\phi}_p + \hat{\phi}_{p'} - \hat{\phi}_k - \hat{\phi}_{k'} \right). \end{aligned} \quad (118)$$

Therefore independently of the underlying NEBE the transport coefficients can be calculated by any classical method of choice. However as we have shown, differences arise in the thermodynamical integrals and the EoS.

A.2 Grad's method of moments

Here we discuss an alternative method to calculate the transport coefficients. This method, originally due to Grad [54] leads not only slightly different transport coefficients but also to different equations of motion for the dissipative quantities. This is due to the different choice of parameters for the non-equilibrium distribution function and the particularities of the method, see for example Refs. [38, 40, 41, 54, 55, 56, 57, 58]. for more details.

Here we follow Refs.[57,58] and introduce the following irreducible tensor moment for the deviation from local q -equilibrium,

$$\tilde{\rho}_r^{\langle \mu_1 \dots \mu_\ell \rangle} = \int dK E_k^r k^{\langle \mu_1} \dots k^{\mu_\ell \rangle} \delta \tilde{f}_k, \quad (119)$$

where the index ℓ indicates the rank of the tensor such that $\ell = 0$ corresponds to the scalar $\tilde{\rho}_r$, with power r of the energy E_k . Similarly one can introduce irreducible moments involving the solution of the other NEBE from Eq. (106), i.e., $\hat{\rho}_r^{\langle \mu_1 \dots \mu_\ell \rangle} = \int dK E_k^r k^{\langle \mu_1} \dots k^{\mu_\ell \rangle} \delta \hat{f}_k$. The generalized irreducible moments from Eq. (119) are identified with the dissipative quantities via the following relations,

$$\begin{aligned} \delta n &\equiv \tilde{\rho}_{(1)} = 0, \quad \delta e \equiv \tilde{\rho}_{(2)} = 0, \quad \Pi = -\frac{m^2}{3} \tilde{\rho}_{(0)}, \\ V^\mu &= \tilde{\rho}_{(0)}^\mu, \quad W^\mu = \tilde{\rho}_{(1)}^\mu, \quad \pi^{\mu\nu} = \tilde{\rho}_{(0)}^{\mu\nu}. \end{aligned} \quad (120)$$

Now, rewriting the NEBE (13) in the following form,

$$\begin{aligned} \frac{d}{d\tau} \delta \tilde{f}_k &= -\frac{d}{d\tau} \tilde{f}_{0k} - E_k^{-1} k_\nu \nabla^\nu \tilde{f}_{0k} \\ &- E_k^{-1} k_\nu \nabla^\nu \delta \tilde{f}_k + E_k^{-1} C[\delta f], \end{aligned} \quad (121)$$

we obtain exact equations for $\frac{d}{d\tau} \tilde{\rho}_r^{\langle \mu_1 \dots \mu_\ell \rangle}$. This method is presented and analysed in great detail in Refs. [57,58].

The main results of Grad's method are the so-called relaxation equations, which determine the time evolution

of $\tilde{\rho}_r^{\langle \mu_1 \dots \mu_\ell \rangle}$ hence also of Π , q^μ , and $\pi^{\mu\nu}$. The relaxation of the dissipative quantities towards their Navier-Stokes values is given with time scales given by the corresponding relaxation times τ_Π , τ_q and τ_π . The calculation of the collision integral actually involves the very same procedure as presented in the previous sections, while the relaxation equations are also given in exactly the same form as in the classical case. Once again the $q \neq 1$ modifications are embedded in the thermodynamical and collision integrals.

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